

*Signali i sustavi*

## Auditorne vježbe 11.

LS&S  
FER - ZESOI

SIS ZESOI *Nastavak ...*

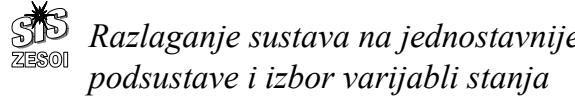
- Izbor varijabli stanja:  
 $x_1(t) = y(t),$   
 $x_2(t) = y'(t) = x_1',$   
 $x_3(t) = y''(t) = x_2'.$
  - To uvrstimo u diferencijalnu jednadžbu  
 $x_3' + 2x_3 + 5x_2 + 6x_1 = u.$

 SIS  
ZESOI Izlazna jednadžba?

$$\mathbf{y} = \mathbf{C} \cdot \mathbf{x} + \mathbf{D} \cdot \mathbf{u},$$

$$y = x_1.$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_G \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_D \cdot u.$$



## DIREKTNA METODA ( NORMALNE VARIJABLE STANJA )

## Zadatak 1.

- Koristeći *direktnu metodu* naći model linearne sustava opisanog diferencijalnom jednadžbom tјekvivalentnom prijenosnom funkcijom.

$$y''' + 2y'' + 5y' + 6y = u$$



- $y''' + 2y'' + 5y' + 6y = u$ .
  - Primjenimo Laplaceovu transformaciju:  
 $s^3Y(s) + 2s^2Y(s) + 5sY(s) + 6Y(s) = U(s)$  (1)
  - (početni uvjeti neka su nula).
  - Nakon izlučivanja  $Y(s)$  imamo:  
 $Y(s) \cdot [s^3 + 2s^2 + 5s + 6] = U(s)$ ,
  - i konačno:
$$H(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^3 + 2s^2 + 5s + 6}.$$
  - To je prijenosna funkcija sustava

 SIS ZESOI Rješenje nastavak ...

- Jednadžbe stanja:  

$$\mathbf{x}' = \mathbf{A} \mathbf{x} + \mathbf{B} \cdot \mathbf{u}$$
  - U našem slučaju:  

$$x_1' = x_2,$$
  

$$x_2' = x_3,$$
  

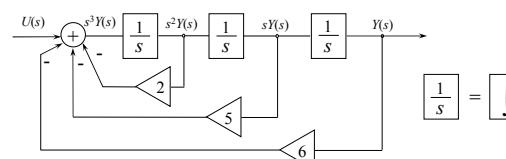
$$x_3' = -6x_1 - 5x_2 - 2x_3 + u$$

 SIS ZESOI *Matrični oblik jed. stanja ..*

- U matričnom obliku, to izgleda ovako:

$$\begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -5 & -2 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

 SIS ZESOI *Simulacijski blok dijagram?*



- $$\bullet \quad \text{iz (1)} \Rightarrow s^3 Y(s) = U(s) - 6Y(s) - 5sY(s) - 2s^2Y(s)$$

 ZESOI *Zadatak*

- Koristeći *direktnu metodu* naći model linearog sustava opisanog diferencijalnom jednadžbom tj. ekvivalentnom prijenosnom funkcijom:

$$H(s) = \frac{s^3 + 2s^2 + 3s + 4}{s^3 + 2s^2 + 5s + 6}$$

- #### ■ Rješenje:

■ Rješenje:

$$H(s) = \frac{B(s)}{A(s)} \quad Y(s) = H(s) \cdot U(s) = \frac{B(s)}{A(s)} \underbrace{U(s)}_{Z(s)} = B(s) \cdot Z(s)$$



## Zadatak 2.

- Najprije realiziramo  $Z(s)$ :

$$Z(s) = \frac{1}{s^3 + 2s^2 + 5s + 6} \cdot U(s).$$

$$z''' + 2z'' + 5z' + 6z = u, \quad (1)$$

- ovo je isti slučaj kao i u prethodnom zadatku!



## Što ćemo s brojnikom $B(s)$ ?

$$Y(s) = (s^3 + 2s^2 + 3s + 4) \cdot Z(s),$$

$$y(t) = z''' + 2z'' + 3z' + 4z, \quad (2)$$

$$z''' + 2z'' + 5z' + 6z = u. \quad (1)$$

- Jednadžbe stanja (iz (1)) su:

$$x_1 = z,$$

$$x_2 = z' \quad x_1' = x_2,$$

$$x_3 = z'' \quad x_2' = x_3,$$

$$x_3' = -6x_1 - 5x_2 - 2x_3 + u.$$



## Rješenje nastavak ...

- Matrični oblik:

$$\begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u,$$

$$y = [-2 \quad -2 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 1 \cdot u.$$

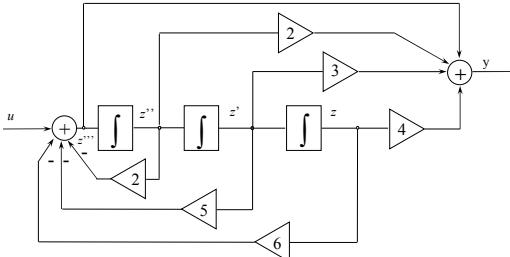


## Simulacijski blok dijagram?

- Nazivnik  $\rightarrow z''' + 2z'' + 5z' + 6z = u$ .

Ovo znamo realizirati (prethodni zadatak).

- Brojnik  $y(t) = z''' + 2z'' + 3z' + 4z$ .



## Kaskadna realizacija

- Faktoriziramo razlomak

$$H(s) = \frac{s+2}{s+3} \cdot \frac{s+1}{s+4} = H_1(s) \cdot H_2(s),$$

$$H_1(s) = \frac{s+2}{s+3},$$

$$Y_1(s) = H_1(s) \cdot U(s), \quad X_1(s) = \frac{s+2}{s+3} U(s),$$

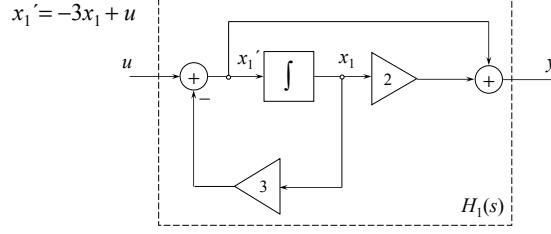
$$X_1(s) = \frac{U(s)}{s+3} \Rightarrow x_1' = -3x_1 + u, \quad (1)$$

$$Y_1(s) = (s+2) \cdot X_1(s) \Rightarrow y_1 = x_1' + 2x_1 = -x_1 + u.$$

$$Y_1(s) = (s+2) \cdot X_1(s) \Rightarrow y_1 = x_1' + 2x_1 = -x_1 + u.$$



## Kaskadna realizacija



$$Y(s) = H_2(s) \cdot Y_1(s) = \frac{s+1}{s+4} Y_1(s), \quad X_2(s) = \frac{Y_1(s)}{s+4} \Rightarrow x_2' = -4x_2 + y_1 = -x_1 - 4x_2 + u. \quad (2)$$



## Rješenje nastavak ...

- Izlazna jednadžba

$$y = z''' + 2z'' + 3z' + 4z, \quad (2)$$

$$= (-6x_1 - 5x_2 - 2x_3 + u) + 2x_3 + 3x_2 + 4x_1,$$

$$= -2x_1 - 2x_2 + u.$$



## Kaskadna realizacija (Iterativne varijable stanja)

### Zadatak 3.

Za sustav zadan prijenosnom funkcijom nacrtati model i napisati jednadžbe stanja *kaskadnom metodom*.

$$H(s) = \frac{(s+2)(s+1)}{(s+3)(s+4)}.$$

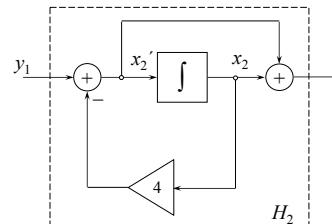
$$\text{Rješenje: } U(s) \xrightarrow{H(s)} Y(s)$$

$$U(s) \xrightarrow{H_1(s)} Y_1(s) \xrightarrow{H_2(s)} Y(s)$$



## Kaskadna realizacija

$$Y(s) = (s+1)X_2(s) \Rightarrow y = x_2' + x_2 = -x_1 - 3x_2 + u.$$





## ZESOI Kaskadna realizacija

- Jednadžbe stanja:

$$\begin{aligned}x_1' &= -3x_1 + u, \\x_2' &= -x_1 - 4x_2 + u.\end{aligned}$$

$y = [-1 \quad -3] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 1 \cdot u.$

Tipična donja trokutasta matrica  
(kod kaskadne realizacije)



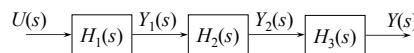
## ZESOI Kaskadna realizacija

### Zadatak 4.

Za sustav zadan prijenosnom funkcijom nacrtati model i napisati jednadžbe stanja kaskadnom metodom.

$$H(s) = \frac{(s+1)(s+2)s}{(s+3)(s+4)(s^2+2s+2)} = \frac{s+1}{s+3} \cdot \frac{s}{s+4} \cdot \frac{s+2}{s^2+2s+2}.$$

ne razbija se dalje, da ne dobijemo imaginarni koeficijente ( $-1 \pm j$ )



## ZESOI Kaskadna realizacija

- $H_2$

$$\begin{aligned}Y_2(s) &= H_2(s) \cdot Y_1(s), \\&= \frac{s}{s+4} Y_1(s),\end{aligned}$$

$$X_2(s) = \frac{Y_1(s)}{s+4} \Rightarrow x_2' = -4x_2 + y_1 = -2x_1 - 4x_2 + u,$$

$$Y_2(s) = s \cdot X_2(s) \Rightarrow y_2 = x_2' = -2x_1 - 4x_2 + u.$$



## ZESOI Kaskadna realizacija

### $H_3$

$$Y(s) = H_3(s) \cdot Y_2(s),$$

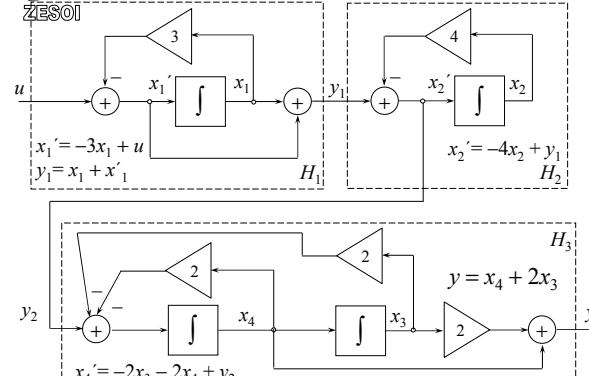
$$= \frac{s+2}{s^2+2s+2} Y_2(s),$$

$$\begin{aligned}Z(s) &= \frac{Y_2(s)}{s^2+2s+2} \Rightarrow z'' + 2z' + 2z = y_2, \\x_3 &= z, \\x_4 &= z' = x_3', \\x_4' &= z'' = -2x_3 - 2x_4 + y_2, \\&= -2x_1 - 4x_2 - 2x_3 - 2x_4 + u.\end{aligned}$$

realizirati direktnom metodom



## ZESOI Kaskadna realizacija



## ZESOI Paralelna realizacija (Kanonske varijable stanja)

$$H(s) = \frac{b_m s^m + \dots + b_0}{a_n s^n + \dots + a_0} = d_0 + \frac{c_1}{s - s_1} + \dots + \frac{c_n}{s - s_n}.$$

Rastav na parcijalne razlomke.

$s_i, \quad i = 1, \dots, n$  - jednostručni realni polovi.

$$\begin{aligned}d_0 &= \lim_{s \rightarrow \infty} H(s) & c_k &= (s - s_k) H(s) \Big|_{s=s_k}, \\Y(s) &= H(s) U(s) = d_0 \cdot U(s) + \sum_{k=1}^n \frac{c_k}{s - s_k} U(s).\end{aligned}$$



## ZESOI Kaskadna realizacija

### $H_1$

$$\begin{aligned}Y_1(s) &= H_1(s) \cdot U(s), \\&= \frac{s+1}{s+3} U(s),\end{aligned}$$

$$X_1(s) = \frac{U(s)}{s+3} \Rightarrow x_1' = -3x_1 + u,$$

$$Y_1(s) = (s+1) \cdot X_1(s) \Rightarrow y_1 = x_1' + x_1 = -2x_1 + u.$$



## ZESOI Kaskadna realizacija

$$Y(s) = (s+2) \cdot Z(s) \Rightarrow y = z' + 2z = x_4 + 2x_3.$$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 & 0 \\ -2 & -4 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -4 & -2 & -2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \cdot u,$$

$$y = [0 \quad 0 \quad 2 \quad 1] \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + 0 \cdot u.$$

Trokutasti oblik "pokvaren" zbog direktnе realizacije sekciјe II reda



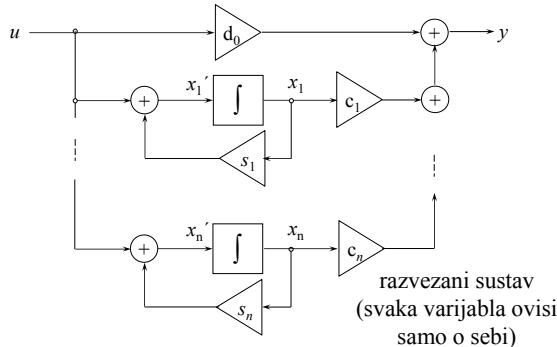
## ZESOI Paralelna realizacija

$$X_k(s) = \frac{U(s)}{s - s_k} \Rightarrow x_k' = s_k \cdot x_k + u,$$

$$Y(s) = d_0 U(s) + \sum_{k=1}^n c_k X_k(s) \Rightarrow y = d_0 u + \sum_{k=1}^n c_k x_k.$$



## ZESOI Paralelna realizacija



## ZESOI Paralelna realizacija, izbor varijabli stanja

$$Y(s) = -6 \cdot \frac{1}{(s+1)^2} U(s) - 5 \cdot \frac{1}{(s+1)} U(s) + 6 \cdot \frac{1}{s} U(s) - \frac{1}{(s+2)} U(s)$$

$$X_1(s) = \frac{U(s)}{(s+1)^2} = \frac{X_2(s)}{s+1} \Rightarrow x_1' = -x_1 + x_2,$$

$$X_2(s) = \frac{U(s)}{s+1} \Rightarrow x_2' = -x_2 + u,$$

$$X_3(s) = \frac{U(s)}{s} \Rightarrow x_3' = u,$$

$$X_4(s) = \frac{U(s)}{s+2} \Rightarrow x_4' = -2x_4 + u.$$



## ZESOI Jordanov blok u općem slučaju:

$$\begin{bmatrix} p & 1 & 0 & 0 & 0 \\ 0 & p & 1 & 0 & 0 \\ 0 & 0 & p & 1 & 0 \\ 0 & 0 & 0 & p & 1 \\ 0 & 0 & 0 & 0 & p \end{bmatrix} n=5$$



## ZESOI Paralelna realizacija

### Zadatak 5

Nacrtati simulacijski dijagram i napisati stanja pomoću kanonskih varijabli (paralelna realizacija)

$$H(s) = \frac{s^2 + 7s + 12}{s(s+1)^2(s+2)}.$$

Rješenje:

$$H(s) = d_0 + \frac{c_{11}}{(s+1)^2} + \frac{c_{12}}{(s+1)} + \frac{c_{21}}{s} + \frac{c_{31}}{(s+2)},$$

uočiti rastav višestrukog pola!!!



## ZESOI Paralelna realizacija

### Višestruki polovi

$$c_{ij} = \frac{1}{(j-1)!} \cdot \frac{d^{j-1}}{ds^{j-1}} [(s-s_i)^{\varphi_j} H(s)]|_{s=s_i},$$

$$c_{11} = (s+1)^2 \cdot H(s)|_{s=-1} = (s+1)^2 \cdot \frac{s^2 + 7s + 12}{s(s+1)^2(s+2)}|_{s=-1} = \dots = -6,$$

$$c_{12} = 1 \cdot \frac{d}{ds} \left[ \frac{s^2 + 7s + 12}{s(s+2)} \right]_{s=-1} = \frac{(2s+7)s(s+2) - (s^2 + 7s + 12)(2s+2)}{[s(s+2)]^2}|_{s=-1} = \dots = -5,$$

$$c_{21} = sH(s)|_{s=0} = 6, \quad c_{31} = (s+2)H(s)|_{s=-2} = -1.$$



## ZESOI Paralelna realizacija

### Jednadžbe stanja: Jordanov blok, -1 višestruki korjen

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot u.$$



## ZESOI Paralelna realizacija

### Izlazna jednadžba:

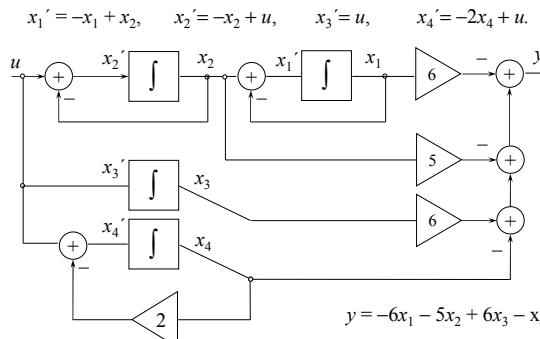
$$y = [-6 \ -5 \ 6 \ -1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + 0 \cdot u.$$

Za jednostrukе polove u matrici A ostaju samo dijagonalni elementi.

Za n-strike polove  $p$  javlja se Jordanov blok ( $n > 1$ ).



## ZESOI Blok dijagram (realizacija):



## ZESOI Zadatak 6., paralelna realizacija

$$H(s) = \frac{(s+2) \cdot (s+1)}{(s+3) \cdot (s^2 + 4s + 5)},$$

$s_{1,2} = -2 \pm j \Rightarrow$  konjugirano kompleksna rješenja

$$H(s) = d_0 + \frac{C_1}{s+3} + \frac{C_2 \cdot s + C_3}{s^2 + 4s + 5},$$

$$d_0 = \lim_{s \rightarrow \infty} H(s) = 0,$$

$$C_1 = (s+3)H(s)|_{s=-3} = 1.$$



## nastavak

- $C_2, C_3 = ?$
- Metoda jednakih koeficijenata.

$$H(s) = \frac{(s+2)(s+1)}{(s+3)(s^2+4s+5)} = \frac{1}{s+3} + \frac{C_2 \cdot s + C_3}{s^2+4s+5},$$

$$\frac{s^2+3s+2}{(s+3)(s^2+4s+5)}$$



## nastavak

- Izjednačimo brojnine

$$s^2 + 3s + 2 = s^2(1 + C_2) + s(4 + 3C_2 + C_3) + (5 + 3C_3),$$

$$1 + C_2 = 1 \Rightarrow C_2 = 0,$$

$$5 + 3C_3 = 2 \Rightarrow C_3 = -1,$$

$$4 + 3C_2 + C_3 = 3,$$

$$4 + 3 \cdot 0 - 1 = 3,$$

$$3 = 3.$$



## nastavak

$$H(s) = \frac{1}{s+3} - \frac{1}{s^2+4s+5},$$

$$Y(s) = H(s) \cdot U(s) = \frac{1}{s+3} U(s) - \frac{1}{s^2+4s+5} U(s),$$

$$X_1(s) = \frac{U(s)}{s+3} \Rightarrow x_1' = -3x_1 + u,$$

$$X_2(s) = \frac{U(s)}{s^2+4s+5} \Rightarrow x_2' + 4x_2 + 5x_2 = u,$$

$$x_2' = x_3,$$

$$x_3' + 4x_3 + 5x_2 = u,$$

$$x_3' = -5x_2 - 4x_3 + u.$$



## nastavak

$$Y(s) = x_1(s) - x_2(s),$$

$$y = x_1 - x_2.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u.$$

Matrica A nije dijagonalna i nema samo Jordanove blokove.

$$y = [1 \ -1 \ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0 \cdot u.$$



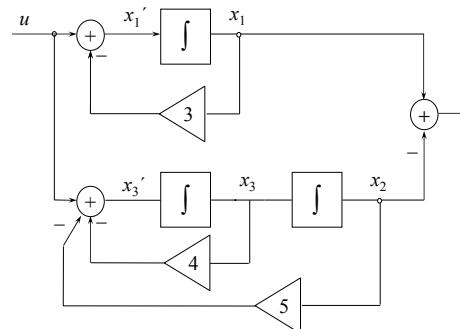
## nastavak

$$x_1' = -3x_1 + u,$$

$$x_2' = x_3,$$

$$y = x_1 - x_2.$$

$$x_3' = -5x_2 - 4x_3 + u,$$



## Zadatak 7., paralelna realizacija

$$H(s) = \frac{s(s+1)}{(s+2)(s+1)}$$

- $s+1$  se ne smije pokratiti u slučaju kada se traže varijable stanja

- par pol/nula postoji u sustavu, utječe na njegovo vladanje, ali je nevidljiv s ulazno-izlaznih stezaljki



## Zadatak 7., paralelna realizacija

$$H(s) = \frac{s(s+1)}{(s+2)(s+1)}$$

Rješenje:

$$H(s) = d_0 + \frac{C_1}{s+2} + \frac{C_2}{s+1}$$

$$\left. \begin{array}{rcl} d_0 & = & 1 \\ C_1 & = & -2 \\ C_2 & = & 0 \end{array} \right\} H(s) = 1 - 2 \cdot \frac{1}{s+2} + 0 \cdot \frac{1}{s+1}$$



## nastavak

$$Y(s) = U(s) - 2 \cdot \frac{U(s)}{s+2} + 0 \cdot \frac{U(s)}{s+1}$$

$$x_1' = -2x_1 + u$$

$$x_2' = -x_2 + u$$

$$y = -2x_1 + 0 \cdot x_2 + u$$



## nastavak

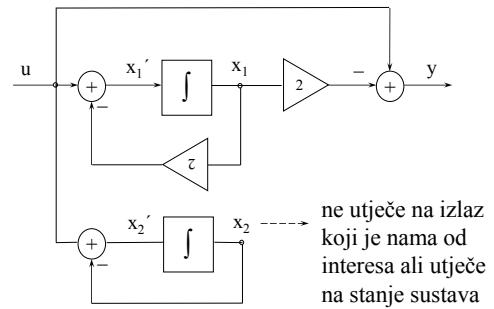
Matrični oblik:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = [-2 \ 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 1 \cdot u$$

$$Y(s) = U(s) - 2X_1(s) + 0 \cdot X_2(s)$$

$$y = -2x_1 + 0 \cdot x_2 + u$$



ne utječe na izlaz  
koji je nama od  
interesa ali utječe  
na stanje sustava