

## Signali i sustavi

### AUDITORNE VJEŽBE 5

LS&S  
FER – ZESOI

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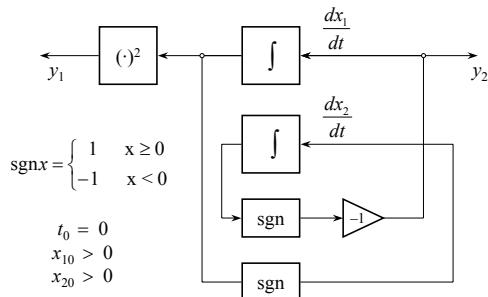


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Zadatak 1.




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Rješenje:

$$y_1 = \left[ \int_0^t y_2(\tau) d\tau + x_{10} \right]^2,$$

$$y_2 = -\text{sign} \left[ \int_0^t \text{sign} \left( \int_0^\tau y_2(\lambda) d\lambda + x_{10} \right) d\tau + x_{20} \right].$$

- Bez sumnje, složena zadaća za analitičko rješavanje!

Jednadžbe stanja:

$$\frac{dx_1}{dt} = -\text{sgnx}_2,$$

$$\frac{dx_2}{dt} = \text{sgn} x_1.$$

Izlazne jednadžbe:

$$y_1 = x_1^2,$$

$$y_2 = -\text{sgnx}_2.$$

- Problem je jednostavnije rješiti pomoću varijabli stanja (izabrati  $x_1$  i  $x_2$ ).

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Problem čemo riješiti geometrijski u ravnini stanja!

$$\begin{array}{c|c} \frac{dx_1}{dt} = -1, \frac{dx_2}{dt} = -1 & \frac{dx_1}{dt} = -1, \frac{dx_2}{dt} = 1 \\ \hline \frac{dx_1}{dt} = 1, \frac{dx_2}{dt} = -1 & \frac{dx_1}{dt} = 1, \frac{dx_2}{dt} = 1 \end{array}$$

- Kako  $\frac{dx_1}{dt}$  i  $\frac{dx_2}{dt}$  poprimaju jednu od dvije vrijednosti  $\{-1, 1\}$ , slijedi:

2. i 4. kvadrant

$$\frac{dx_2}{dx_1} = 1.$$

1. i 3. kvadrant

$$\frac{dx_2}{dx_1} = -1.$$

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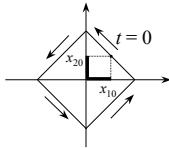
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ZESOI

$$\frac{dx_1}{dt} = -1, \frac{dx_2}{dt} = 1 \Rightarrow \frac{dx_2}{dx_1} = -1.$$

- Ovu činjenicu čemo iskoristiti u crtanju trajektorije varijabli stanja
- Ograničimo se na 1. kvadrant ( $x_{10}, x_{20} > 0$ )
- Dakle, trajektorija je pravac!

Kako će se mijenjati stanje?



Imamo periodičko kruženje!

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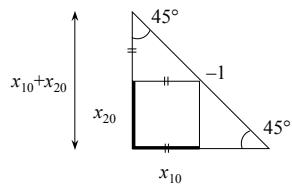
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ZESOI

- Nacrtajmo još jednom prvi kvadrant.
- Nagib pravca je  $-1$ , to znači  $45^\circ$ .
- Onda su označene dužine jednake (na slici ||) !!
- Nadalje, očigledno je:




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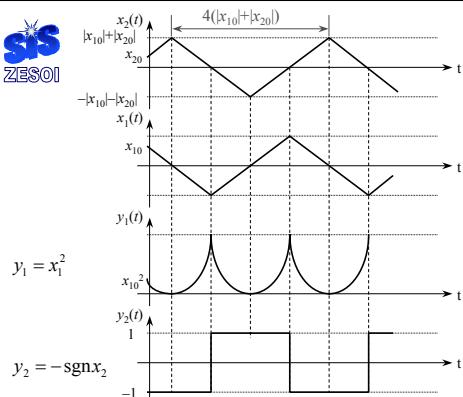
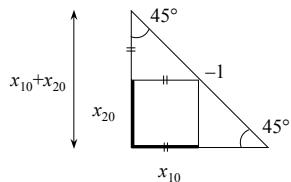
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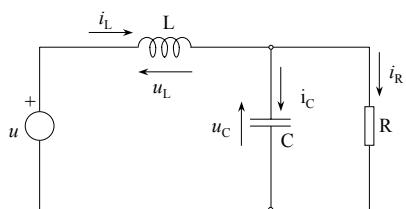
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- Iz slike zaključujemo da  $x_1$  i  $x_2$  imaju svoj maksimum koji je  $|x_{10}| + |x_{20}|$ , dok je minimum  $-|x_{10}| - |x_{20}|$ .
- Nadalje, kada jedna varijabla stanja postiže maksimum (minimum) druga prolazi kroz nulu.
- Oba stanja se mijenjaju po periodičnim funkcijama perioda  $4(|x_{10}| + |x_{20}|)$  – što će biti jasnije iz narednih slika.



### Zadatak 2.

Napisati jednadžbe stanja i izlazne jednadžbe za električnu mrežu priказанu slikom.  $u$  je ulaz u sustav, a  $i_R$  izlaz iz sustava.





## Odabir varijabli stanja (sistemi s memorijskim elementima)

- L i C su memorijski elementi.

$$u_L = L \frac{di_L}{dt} \Rightarrow \frac{di_L}{dt} = \frac{u_L}{L}$$

$$i_C = C \frac{du_C}{dt} \Rightarrow \frac{du_C}{dt} = \frac{i_C}{C}$$

- Varijable stanja električne mreže su  $i_L, u_C$ .
- Za jednadžbe stanja treba naći  $\frac{di_L}{dt} = \dots, \frac{du_C}{dt} = \dots$

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## Nacín rješavanja

- Zadana električna mreža je linearna.
- Koristit će se teorem superpozicije.
- Doprinos pojedinog „aktivnog“ elementa mreže određuje se tako da se „isključe“ sve preostale „aktivne“ komponente.
- „Isključiti“, to znači: C, u → kratko spojiti,  
L, i → odspojiti,
- gdje su u, i → nezavisni naponski ili strujni izvori.
- Ukupni odziv jednak je sumi doprinosa pojedinih aktivnih elemenata.

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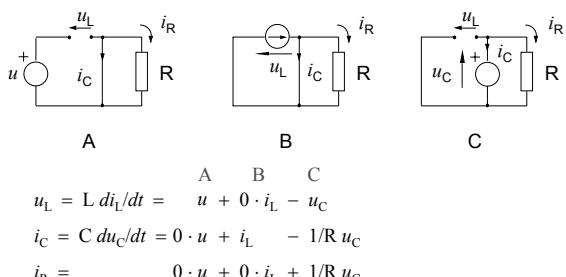
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Slučaj	A	B	C
Uključen	u	L	C
Isključen	L, C	u, C	u, L




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- Ako podijelimo jednadžbe s L, odnosno C dobijemo:

$$\frac{di_L}{dt} = \frac{1}{L}u - \frac{1}{L}u_C,$$

$$\frac{du_C}{dt} = \frac{1}{C}i_L - \frac{1}{RC}u_C,$$

- što su željene jednadžbe stanja, uz već poznatu izlaznu jednadžbu:

$$i_R = \frac{1}{R}u_C.$$

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- U matričnom obliku, to izgleda ovako:

$$\begin{bmatrix} \frac{di_L}{dt} \\ \frac{du_C}{dt} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} i_L \\ u_C \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u,$$

$$i_R = \begin{bmatrix} 0 & \frac{1}{R} \end{bmatrix} \begin{bmatrix} i_L \\ u_C \end{bmatrix} + 0 \cdot u.$$

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## Razlaganje sustava na jednostavnije podsustave i izbor varijabli stanja

### DIREKTNA METODA ( NORMALNE VARIJABLE STANJA )

#### Zadatak 1.

- Koristeći *direktnu metodu* naći model linearog sustava opisanog diferencijalnom jednadžbom tj. ekvivalentnom prijenosnom funkcijom.

$$y''' + 2y'' + 5y' + 6y = u.$$

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### Rješenje :

- $y''' + 2y'' + 5y' + 6y = u.$
- Primjenimo Laplaceovu transformaciju:  
 $s^3Y(s) + 2s^2Y(s) + 5sY(s) + 6Y(s) = U(s)$  (1)
- (početni uvjeti neka su nula).
- Nakon izlučivanja  $Y(s)$  imamo:  
 $Y(s) \cdot [s^3 + 2s^2 + 5s + 6] = U(s),$
- i konačno:  
$$H(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^3 + 2s^2 + 5s + 6}.$$
- To je prijenosna funkcija sustava.

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### Nastavak ...

- Izbor varijabli stanja:  
 $x_1(t) = y(t),$   
 $x_2(t) = y'(t) = x_1',$   
 $x_3(t) = y''(t) = x_2'.$
- To uvrstimo u diferencijalnu jednadžbu  
 $x_3' + 2x_3 + 5x_2 + 6x_1 = u.$

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### Rješenje nastavak ...

- Jednadžbe stanja:  
 $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{B} \cdot \mathbf{u}.$
- U našem slučaju:  
 $x_1' = x_2,$   
 $x_2' = x_3,$   
 $x_3' = -6x_1 - 5x_2 - 2x_3 + u.$

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### Matrični oblik jed. stanja ...

- U matričnom obliku, to izgleda ovako:

$$\begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -5 & -2 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{B}} u.$$

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### Izlazna jednadžba?

$$\mathbf{y} = \mathbf{C} \cdot \mathbf{x} + \mathbf{D} \cdot \mathbf{u},$$

$$y = x_1.$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{\mathbf{C}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{\mathbf{D}} \cdot u.$$

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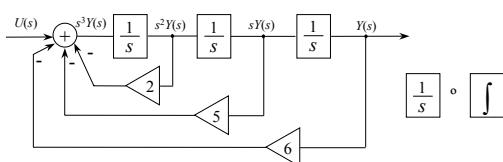
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### Simulacijski blok dijagram?



- iz (1)  $\Rightarrow s^3Y(s) = U(s) - 6Y(s) - 5sY(s) - 2s^2Y(s).$

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## Zadatak 2.

- Koristeći *direktnu metodu* naći model linearog sustava opisanog diferencijalnom jednadžbom tj. ekvivalentnom prijenosnom funkcijom:

$$H(s) = \frac{s^3 + 2s^2 + 3s + 4}{s^3 + 2s^2 + 5s + 6} = \frac{B(s)}{A(s)}.$$

- Rješenje:

$$\begin{aligned} H(s) &= \frac{B(s)}{A(s)} & Y(s) &= H(s) \cdot U(s) = \underbrace{\frac{B(s)}{A(s)}}_{Z(s)} \underbrace{U(s)}, \\ &&&= B(s) \cdot Z(s). \end{aligned}$$

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## Zadatak 2.

- Najprije realiziramo  $Z(s)$ :

$$Z(s) = \frac{1}{s^3 + 2s^2 + 5s + 6} \cdot U(s).$$

$$z''' + 2z'' + 5z' + 6z = u, \quad (1)$$

- ovo je isti slučaj kao i u prethodnom zadatku!

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## Što ćemo s brojnikom $B(s)$ ?

$$Y(s) = (s^3 + 2s^2 + 3s + 4) \cdot Z(s),$$

$$y(t) = z''' + 2z'' + 3z' + 4z, \quad (2)$$

$$z''' + 2z'' + 5z' + 6z = u. \quad (1)$$

- Jednadžbe stanja ( iz (1) ) su :

$$x_1 = z,$$

$$x_2 = z' \quad x_1' = x_2,$$

$$x_3 = z'' \quad x_2' = x_3,$$

$$x_3' = -6x_1 - 5x_2 - 2x_3 + u.$$

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### Rješenje nastavak ...

- Izlazna jednadžba

$$\begin{aligned} y &= z''' + 2z'' + 3z' + 4z, \quad (2) \\ &= (-6x_1 - 5x_2 - 2x_3 + u) + 2x_3 + 3x_2 + 4x_1, \\ &= -2x_1 - 2x_2 + u. \end{aligned}$$

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### Rješenje nastavak ...

- Matrični oblik:

$$\begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u,$$

$$y = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 1 \cdot u.$$

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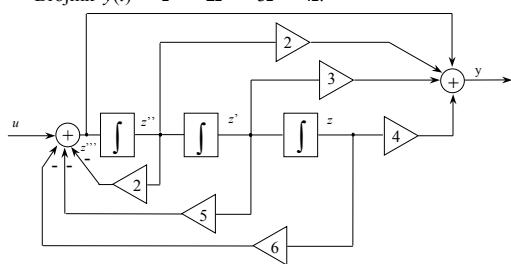


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### Simulacijski blok dijagram?

- Nazivnik  $\rightarrow z''' + 2z'' + 5z' + 6z = u$ .
  - Ovo znamo realizirati (prethodni zadatak).
- Brojnik  $y(t) = z''' + 2z'' + 3z' + 4z$ .




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