Fourierova transformacija

Fourierova transformacija kontinuiranog signala

Definicija:

$$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi Ft} dt$$

$$x(t) = \int_{-\infty}^{\infty} X(F) e^{j2\pi Ft} dF$$

Lema o reciprocitetu

- Ako je X(F) Fourierova transformacija signala x(t) tada je Fourierova transformacija signala X(t), signal x(-F)

$$x(t) \xleftrightarrow[\text{Transformacija}]{\text{Fourierova}} X(F)$$

$$X(t) \xleftrightarrow[\text{Transformacija}]{\text{Fourierova}} x(-F)$$

FT – Diracove δ funkcije

$$X(F) = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft} dt = 1$$

- Prema lemi o reciprocitetu vrijedi

$$\delta(F) = \int_{-\infty}^{\infty} e^{-j2\pi ft} dt$$

FT – sin(t)

- Poznato je da vrijedi $\sin t = \frac{e^{jt} - e^{-jt}}{2j}$

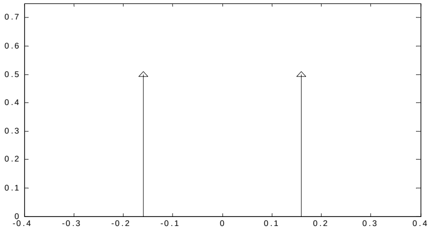
$$X(F) = \int_{-\infty}^{\infty} \sin(t) e^{-j2\pi Ft} dt = \int_{-\infty}^{\infty} \frac{e^{jt} - e^{-jt}}{2j} e^{-j2\pi Ft} dt =$$

$$= \frac{1}{2j} \int_{-\infty}^{\infty} e^{-j2\pi\left(F - \frac{1}{2\pi}\right)t} dt - \frac{1}{2j} \int_{-\infty}^{\infty} e^{-j2\pi\left(F + \frac{1}{2\pi}\right)t} dt =$$

$$= \frac{1}{2j}[\delta(F - \frac{1}{2\pi}) - \delta(F + \frac{1}{2\pi})]$$

FT – sin(t)

- Konačno amplitudni spektar izgleda $|X(F)| = \frac{1}{2}[\delta(F - \frac{1}{2\pi}) + \delta(F + \frac{1}{2\pi})]$



FT – 10 perioda funkcije sin(t)

$$X(F) = \int_{0}^{20\pi} \sin(t) e^{-j2\pi Ft} dt = \int_{0}^{20\pi} \sin(t)[(\cos(2\pi Ft) - j\sin(2\pi Ft)] dt =$$

$$= \int_{0}^{20\pi} [\sin(t)\cos(2\pi Ft) - j\sin(t)\sin(2\pi Ft)] dt =$$

$$= \int_{0}^{20\pi} \sin(t)\cos(2\pi Ft) dt - j\int_{0}^{20\pi} \sin(t)\sin(2\pi Ft) dt =$$

$$= \frac{1}{2}\int_{0}^{20\pi} [\sin((1+2\pi F)t) + \sin((1-2\pi F)t)] dt + \frac{1}{2} j \int_{0}^{20\pi} [\cos((1+2\pi F)t) - \cos((1-2\pi F)t)] dt =$$

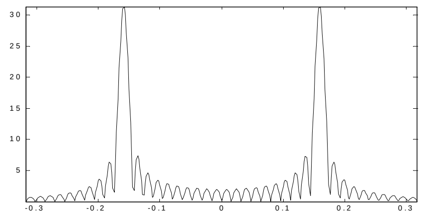
$$= \frac{1 - \cos(40\pi^2 F)}{1 - 4\pi^2 F^2} + j\frac{\sin(40\pi^2 F)}{1 - 4\pi^2 F^2}$$

FT – 10 perioda funkcije sin(t)

- Konačno amplitudni spektar izgleda

$$|X(F)| = \left|\frac{2\sin(20\pi^2 F)}{1 - 4\pi^2 F^2}\right| = \left|\frac{\sin(20\pi^2 F)}{1 - 2\pi F} + \frac{\sin(20\pi^2 F)}{1 + 2\pi F}\right| =$$

$$= \left|\frac{\sin[20\pi^2(F + \frac{1}{2\pi})]}{2\pi(F + \frac{1}{2\pi})} - \frac{\sin[20\pi^2(F - \frac{1}{2\pi})]}{2\pi(F - \frac{1}{2\pi})}\right|$$



FT – 1 periode funkcije sin(t)

- Koristimo teorem o konvoluciji

$$X(F) = \int_{-\pi}^{\pi} e^{-j2\pi Ft} dt = 2\pi\frac{\sin 2\pi^2 F}{2\pi^2 F}$$ Spektar pravokutnog impulsa

$$X(F) = \frac{1}{2j}[\delta(F - \frac{1}{2\pi}) - \delta(F + \frac{1}{2\pi})]$$ Spektar beskonačnog sinusa

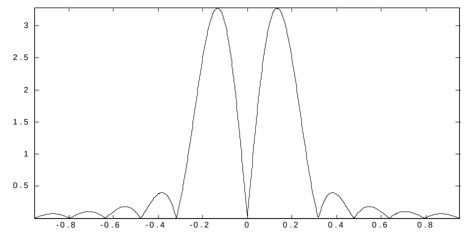
$$X(F) = \int_{-\infty}^{\infty} 2\pi\frac{\sin 2\pi^2\tau}{2\pi^2\tau}\frac{1}{2j}[\delta(F - \tau + \frac{1}{2\pi}) - \delta(F - \tau - \frac{1}{2\pi})] d\tau =$$

$$= \frac{\sin 2\pi^2(F + \frac{1}{2\pi})}{j2\pi(F + \frac{1}{2\pi})} - \frac{\sin 2\pi^2(F - \frac{1}{2\pi})}{j2\pi(F - \frac{1}{2\pi})}$$

FT – 1 periode funkcije sin(t)

- Konačno amplitudni spektar izgleda

$$|X(F)| = \left| \frac{\sin[2\pi^2(F+\frac{1}{2\pi})]}{2\pi(F+\frac{1}{2\pi})} - \frac{\sin[2\pi^2(F-\frac{1}{2\pi})]}{2\pi(F-\frac{1}{2\pi})} \right|$$



Svojstva Fourierove transformacije diskretnih signala

prije je izvedena transformacija za aperiodični diskretni signal (DTFT)

$$X(e^{j\omega}) \equiv F\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$x[n] \equiv F^{-1}\{X(e^{j\omega})\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

ovdje se razmatraju neka svojstva DTFT pogodna u nizu praktičnih primjena

slična razmatranja vrijede i za kontinuirane signale

Svojstva Fourierove transformacije diskretnih signala

svojstvo linearnosti

ako je $X_1(e^{j\omega}) = \mathrm{F}\{x_1[n]\}$ i $X_2(e^{j\omega}) = \mathrm{F}\{x_2[n]\}$ tada je
$X(e^{j\omega}) = \mathrm{F}\{\alpha x_1[n] + \beta x_2[n]\} = \alpha X_1(\omega) + \beta X_2(\omega)$

$$X(e^{j\omega}) \equiv F\{\alpha x_1[n] + \beta x_2[n]\} = \sum_{n=-\infty}^{\infty} (\alpha x_1[n] + \beta x_2[n]) e^{-j\omega n}$$

$$= \alpha \sum_{n=-\infty}^{\infty} x_1[n] e^{-j\omega n} + \beta \sum_{n=-\infty}^{\infty} x_2[n] e^{-j\omega n} =$$

$$= \alpha X_1(e^{j\omega}) + \beta X_2(e^{j\omega})$$

Svojstva Fourierove transformacije diskretnih signala

pomak u vremenskoj domeni

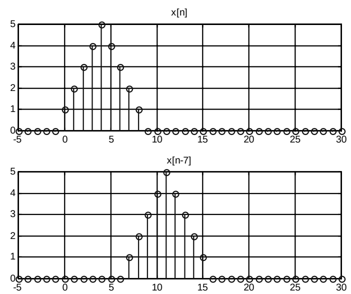
ako je $X(e^{j\omega}) = \mathrm{F}\{x[n]\}$ tada je
$\mathrm{F}\{x[n-k]\} = e^{-j\omega k} X(e^{j\omega})$

$$F\{x[n-k]\} = \sum_{n=-\infty}^{\infty} x[n-k] e^{-j\omega n} =$$

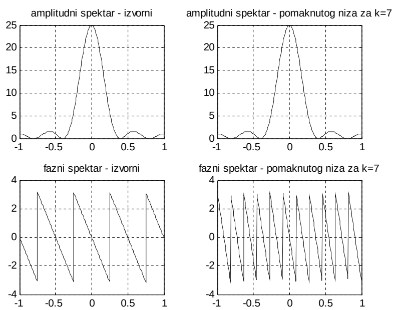
$$= |n-k=m| = \sum_{m=-\infty}^{\infty} x[m] e^{-j\omega(m+k)} =$$

$$= e^{-j\omega k} \sum_{m=-\infty}^{\infty} x[m] e^{-j\omega m} = e^{-j\omega k} X(e^{j\omega})$$

Svojstva Fourierove transformacije diskretnih signala



Svojstva Fourierove transformacije diskretnih signala



Svojstva Fourierove transformacije diskretnih signala

pomak u frekvencijskoj domeni

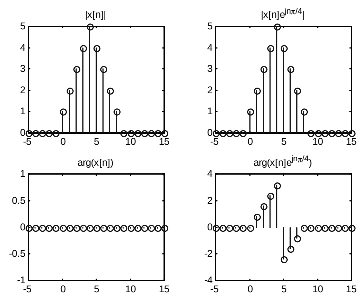
ako je $X(e^{j\omega}) = \mathrm{F}\{x[n]\}$ tada je

$$F\{e^{j\omega_0 n} x[n]\} = X(e^{j(\omega-\omega_0)})$$

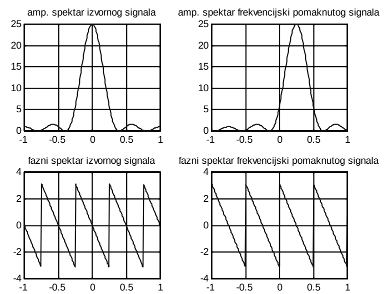
$$F\{e^{j\omega_0 n} x[n]\} = \sum_{n=-\infty}^{\infty} e^{j\omega_0 n} x[n] e^{-j\omega n} =$$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega-\omega_0)n} = X(e^{j(\omega-\omega_0)})$$

Svojstva Fourierove transformacije diskretnih signala



Svojstva Fourierove transformacije diskretnih signala



Svojstva Fourierove transformacije diskretnih signala

konvolucija

ako je $X_1(e^{j\omega}) = \mathrm{F}\{x_1[n]\}$ i $X_2(e^{j\omega}) = \mathrm{F}\{x_2[n]\}$ tada je

$$\mathrm{F}\left\{\sum_{m=-\infty}^{\infty} x_1[m]\cdot x_2[n-m]\right\} = X_1(e^{j\omega})\cdot X_2(e^{j\omega})$$

$$F\left\{\sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m]\right\} = \sum_{n=-\infty}^{\infty}\left[\sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m]\right]e^{-j\omega n} =$$

$$= \sum_{m=-\infty}^{\infty} x_1[m]\left[\sum_{n=-\infty}^{\infty} x_2[n-m]e^{-j\omega n}\right] = \left|n-m=l\right| =$$

Svojstva Fourierove transformacije diskretnih signala

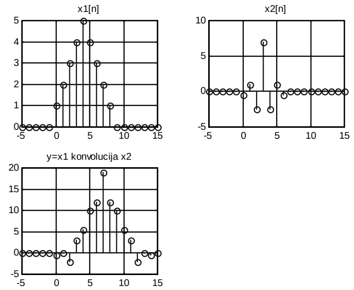
$$= \sum_{m=-\infty}^{\infty} x_1[m]\left[\sum_{l=-\infty}^{\infty} x_2[l]e^{-j\omega(m+l)}\right] =$$

$$= \sum_{m=-\infty}^{\infty} x_1[m]e^{-j\omega m}\left[\sum_{l=-\infty}^{\infty} x_2[l]e^{-j\omega l}\right] =$$

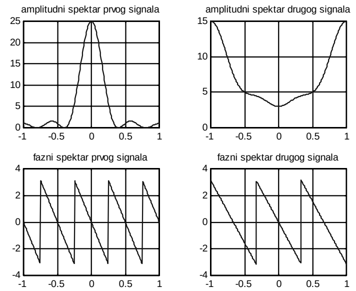
$$= X_2(e^{j\omega})\cdot\sum_{m=-\infty}^{\infty} x_1[m]e^{-j\omega m} =$$

$$= X_2(e^{j\omega})\cdot X_1(e^{j\omega})$$

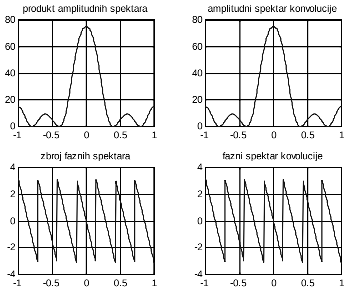
Svojstva Fourierove transformacije diskretnih signala



Svojstva Fourierove transformacije diskretnih signala



Svojstva Fourierove transformacije diskretnih signala



Svojstva Fourierove transformacije diskretnih signala

modulacija

ako je $X_1(e^{j\omega}) = \mathrm{F}\{x_1[n]\}$ i $X_2(e^{j\omega}) = \mathrm{F}\{x_2[n]\}$ tada je

$$\mathrm{F}\{x_1[n]\cdot x_2[n]\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\psi})\cdot X_2(e^{j(\omega-\psi)})d\psi$$

$$\mathrm{y}[\mathrm{n}] = x_1[n]\cdot x_2[n] \Rightarrow Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty}\left(x_1[n]x_2[n]\right)e^{-j\omega n}$$

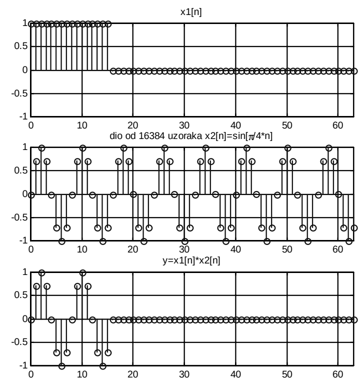
$$x_1[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\psi})e^{j\psi n}d\psi$$

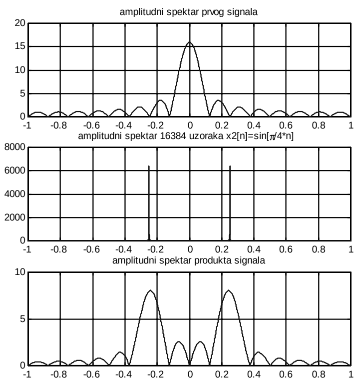
Svojstva Fourierove transformacije diskretnih signala

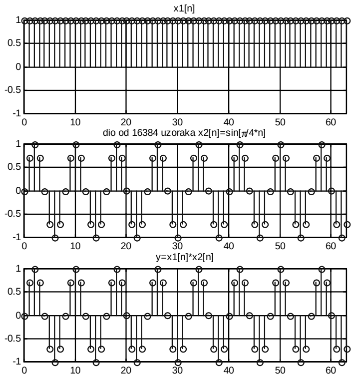
$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\psi})e^{j\psi n}d\psi\right]x_2[n]e^{-j\omega n}$$

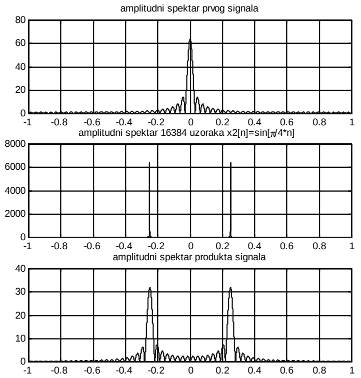
$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\psi})\underbrace{\sum_{n=-\infty}^{\infty} x_2[n]e^{-j(\omega-\psi)n}}_{X_2(e^{j(\omega-\psi)})}d\psi$$

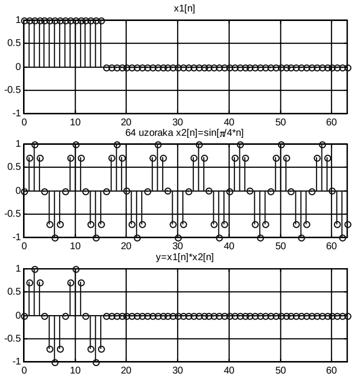
$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\psi})X_2(e^{j(\omega-\psi)})d\psi = \mathrm{F}\{x_1[n]\cdot x_2[n]\}$$

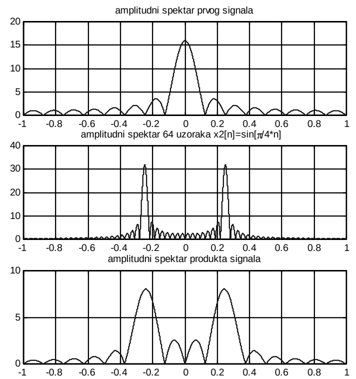


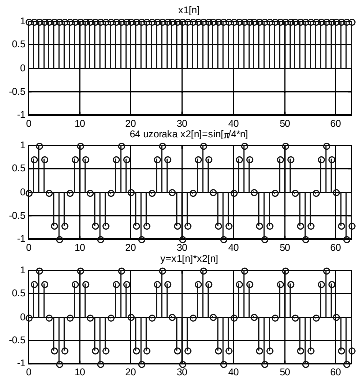


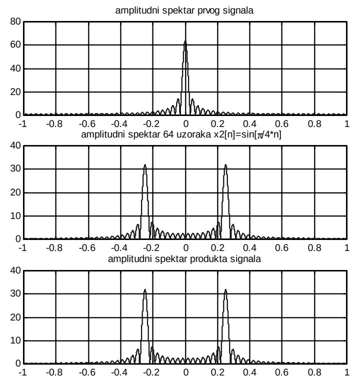


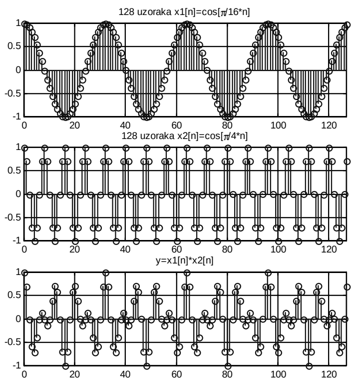












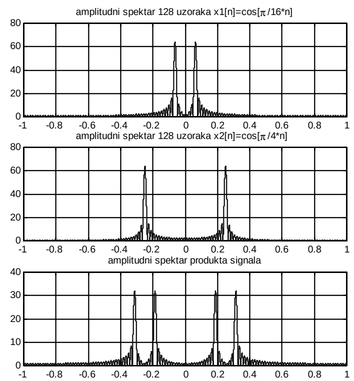
primjer

$$x_1[n] = \cos[\tfrac{\pi}{4} n]$$

$$x_2[n] = \cos[\tfrac{\pi}{16} n]$$

$$x_1[n] \cdot x_2[n] = \cos[\tfrac{\pi}{4} n] \cdot \cos[\tfrac{\pi}{16} n] =$$

$$x_1[n] \cdot x_2[n] = \tfrac{1}{2}\left\{\cos[\tfrac{3\pi}{16} n] + \cos[\tfrac{5\pi}{16} n]\right\} =$$



1/16=
0.0625

1/4=
0.25

3/16=
0,1875

5/16=
0,3125

Svojstva Fourierove transformacije diskretnih signala

inverzija vremenske osi

ako je $X(e^{j\omega}) = \mathrm{F}\{x[n]\}$ tada je $X(e^{-j\omega}) = \mathrm{F}\{x[-n]\}$

$$F\{x[-n]\} = \sum_{n=-\infty}^{\infty} x[-n] e^{-j\omega n} = \left| n = -l \right| =$$

$$= \sum_{l=-\infty}^{\infty} x[l] e^{j\omega l} = X(e^{-j\omega})$$

Svojstva Fourierove transformacije diskretnih signala

za realni $x[n]$ dobivamo:

$$F\{x[-n]\} = X(e^{-j\omega}) = \left| X(e^{-j\omega}) \right| e^{j \arg\{X(e^{-j\omega})\}}$$

$$= \left| X(e^{j\omega}) \right| e^{-j \arg\{X(e^{j\omega})\}}$$